# A THEOREM IN NUMBER THEORY CONNECTED WITH THE BINOMIAL FORMULA.

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The theorem is: *In the expansion of* $(P + Q)^n$, *where* $P$ *and* $Q$ *as well as* $n$ *are integers, there will be a pair of consecutive equal terms when, and only when,* $n \equiv -1 \bmod (P + Q)/\delta$, *where* $\delta$ *is the greatest common divisor of* $P$ *and* $Q$.

Equating two consecutive terms we have

$$\frac{n(n-1)(n-2)(n-3)\cdots(n-k)}{(k+1)!}P^{n-k-1}Q^{k+1} =$$

$$\frac{n(n-1)(n-2)\cdots(n-k-1)}{(k+2)!}P^{n-k-2}Q^{k+2}.$$

Cancelling and multiplying across,

$$P(k+2) = (n-k-1)Q.$$

If now $P = \delta P'$ and $Q = \delta Q'$ where $P'$ and $Q'$ are relatively prime,

$$P'(k+2) = (n-k-1)Q',$$

whence

$$nQ' = 2P' + Q' + k(P' + Q')$$

or

$$(n+1)Q' \equiv 2(P' + Q') \bmod (P' + Q'),$$

that is

$$(n+1)Q' \equiv 0 \bmod (P' + Q'),$$

and since $Q'$ is prime to $P' + Q'$,

$$n + 1 \equiv 0 \bmod (P' + Q').$$

That is

$$n \equiv -1 \bmod (P + Q)/\delta.$$

More generally if $n \equiv -1 \bmod (P + Q)/\delta$ there will be two consecutive terms in the expansion having the ratio $\beta/\alpha$, $\delta$ being the greatest common divisor of $\alpha P$ and $\beta Q$.

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# AN APPLICATION OF PARTIAL DERIVATIVES TO THE ELLIPSE.

By M. O. TRIPP, Yonkers, N. Y.

The object of this article is to show how we may make use of partial derivatives in graphing and discussing the properties of a given ellipse.

Let us take the equation of an ellipse, in rectangular coördinates, in the general form

$$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \quad (h^2 - ab < 0),$$

from which we obtain

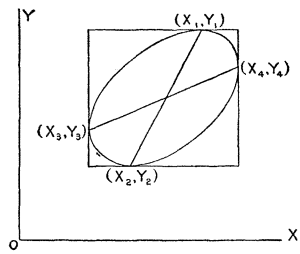
$$\frac{\partial f(x, y)}{\partial x} = 2ax + 2hy + 2g, \quad \text{and} \quad \frac{\partial f(x, y)}{\partial y} = 2hx + 2by + 2f.$$

Making use of the formula

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f(x, y)}{\partial x}}{\dfrac{\partial f(x, y)}{\partial y}}$$

we have, for the slope of the tangent at any point on the ellipse,

$$\frac{dy}{dx} = -\frac{ax + hy + g}{hx + by + f}. \tag{1}$$




At the highest and lowest points of the curve, namely, $(x_1, y_1)$ and $(x_2, y_2)$ respectively, $dy/dx = 0$, since the geometrical tangents at these points are parallel to the $x$-axis. Hence we observe from (1) that the equation

$$ax + hy + g = 0 \tag{2}$$

is satisfied by the coördinates of the points $(x_1, y_1)$ and $(x_2, y_2)$, since the denominator in the right member of (1) cannot become infinite for these values. Thus it follows that (2) is the equation of the diameter joining $(x_1, y_1)$ and $(x_2, y_2)$; it is the equation of the diameter bisecting the chords parallel to the axis of $x$.

At the points farthest to the left and right on the curve, namely, $(x_3, y_3)$ and $(x_4, y_4)$ respectively, $dx/dy = 0$, since the geometrical tangents at these points are perpendicular to the $x$-axis.
Hence

$$hx + by + f = 0, \tag{3}$$

since the numerator of the right member of (1) cannot become infinite for the values considered.

Hence (3) represents the equation of the diameter joining $(x_3, y_3)$ and $(x_4, y_4)$; (3) is the equation of the diameter bisecting chords parallel to the axis of $y$.

It follows that the coördinates of the center of the ellipse are given by solving simultaneously the equations (2) and (3), or what amounts to the same thing, the equations

$$\frac{\partial f(x, y)}{\partial x} = 0 \tag{4}$$

and

$$\frac{\partial f(x, y)}{\partial y} = 0. \tag{5}$$

It is obvious that all of the methods described above may be applied to any conic

$$ax^2 + 2bxy + cy^2 + 2gx + 2fy + c = 0.$$

If any failure results, it is certain that the conic is a special form for which the corresponding concept does not exist. Thus the usual rule for determining whether (6) is or is not a parabola results from the test whether (4) and (5) have or have not a common solution.

The tests for highest and lowest points apply, with obvious restrictions, to any curve whatever.

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## BOOK REVIEWS.

UNDER THE DIRECTION OF W. H. BUSSEY.

*Algebra, First Course.* By EDITH LONG and W. C. BRENKE. The Century Co., New York, 1913. viii+283 pages.

This is the first of a series of texts of correlated mathematics for secondary schools and, in particular, it is the first year's work of a two years' course in algebra and geometry. In its preparation, the authors had in mind two purposes. The first was, in their own words, "to vivify the treatment of algebra by a systematic correlation with geometry." The other was to present the matter in a more simple narrative style than is done in most texts.

To carry out the first purpose, the authors have introduced some of the simpler theorems of geometry and the measurement of geometric quantities as well as some of the fundamental concepts of physics. The proofs of the theorems are in the nature of experimental verification. The introduction of such new matter will undoubtedly stimulate the interest of many students but it is a question if the appeal will be as wide as some have thought. It is evident that a class using this text cannot become as proficient in handling ordinary algebraic operations as is possible with other books. This is more than offset by the gain in the second year's work.

The topics treated are generally well and clearly handled. Where possible, the authors have made use of graphic methods to illustrate the principles. The simple style lends itself well to explanations. In definitions there is danger that it will lead to indistinct or even false conceptions, as when an angle is described as a figure formed by two straight lines starting from the same point.

The authors have requested that teachers using the book omit as little of the text as possible. This request merely points out the fact that the teacher as well as the student will need to be educated in its use.

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